Generalization of global symmetry and its applications to QCD-related physics

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Abstract Symmetry has been a powerful tool to uncover nonperturbative aspects of strongly-coupled Quantum Field Theories (QFTs), including the dynamics of Quantum Chromodynamics (QCD). Recently, the notion of symmetry is drastically extended as people noticed that the essential feature of the conservation law or of the Ward-Takahashi identity is the presence of topological operators. Therefore, it is convenient to define generalized symmetry as the topological operators in studying strongly-coupled QFTs. Here, the development of generalized symmetry will be briefly reviewed including its applications to QCD-related physics.

Introduction

Solving QFTs is a difficult task. For example, although gluons and quarks are natural degrees of freedom describing high-energy behaviors, they do not appear in low-energy physics, and instead, one can only observe the color-singlet hadrons. Our theoretical methods are not yet powerful enough to give controllable analytical computations of such physics, and then it would be natural to ask if one can give any exact statements based on QFTs.

Here, let us pay attention to "symmetry". According to the Noether theorem, symmetry provides a conservation law, and one can obtain fruitful theorems as its outcomes:

- **Selection rules and spectral degeneracy**: The states should form the (projective) representation of the symmetry. Especially when the unbroken symmetry exists, the particle spectrum should follow its representation and the scattering amplitudes obey the selection rule.

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- **Landau's criterion of phases** [Landau, 1937]: When the symmetry-breaking patterns are different, there has to be a phase transition separating those two points as long as the symmetry is preserved.
- **Nambu-Goldstone theorem** [Nambu, 1960, Goldstone, 1961]: When continuous symmetry is spontaneously broken, there have to be massless particles created/annihilated by the broken current.
- **Anomaly matching condition** ['t Hooft, 1980, Frishman et al., 1981]: When the symmetry exists, one can couple the background gauge field but the background gauge invariance may not be maintained. This 't Hooft anomaly is preserved by any local, symmetric perturbations, especially by the renormalization-group (RG) flow.

Of course, symmetry has played a crucial role in the study of strongly-coupled QFTs, including that of QCD, for almost a century, and one might guess that there is nothing new here. Surprisingly, this guess turns out to be false, and people start to find many new "symmetries" that have been missed. More precisely speaking, it turns out that there exists a useful generalization about the notion of symmetry: The symmetry is extended in a way that the above theorems still hold and they are applicable to much wider class of QFTs.

The purpose of this review is to give an idea about generalized symmetry and to provide some examples of its applications. The key message is that symmetry is nothing but the data of topological defect operators in various codimensions. In the following, let us first explain why this is a nice generalization of the conventional symmetry, and then discuss its applications to 4d gauge theories. Unless explicitly stated, we shall restrict our attention to the local, unitary, and relativistic QFTs in Euclidean signature.

Symmetry ⇒ Topological defects

In this section, let us give a motivation to regard the topological defects as generalized symmetries. For this purpose, it would be convenient to remind ourselves of the conventional continuous symmetry. Let us then assume that we have a QFT obtained by the classical action $S[\phi]$ with the fundamental field $\phi(x)$, and $S[\phi]$ is invariant under the continuous symmetry $\phi(x) \mapsto \mathrm{e}^{i\alpha}\phi(x)$: $S[\mathrm{e}^{i\alpha}\phi] = S[\phi]$. Let us make the transformation parameter α spacetime dependent, $\alpha \to \alpha(x)$, then the classical action changes as

$$S[\mathrm{e}^{i\alpha(x)}\phi(x)] = S[\phi(x)] + i\int \partial_\mu \alpha\, j^\mu + O(\alpha^2), \tag{1}$$

where j^μ is the Noether current. When ϕ is the solution of the classical equation of motion, $\int \partial_\mu \alpha\, j^\mu$ should vanish for any infinitesimal $\alpha(x)$, and it derives the Noether conservation law:

$$\partial_\mu j^\mu(x) = 0. \tag{2}$$

So far, everything is classical. The Noether theorem can be extended to the QFT counterpart, and it can be obtained by repeating the above analysis in the path-integral formalism. That is the Ward-Takahashi (WT) identity, which claims that (2) holds as an operator identity:

$$\langle \partial_\mu j^\mu(x) O_1(x_1)\cdots O_n(x_n)\rangle = \sum_{m=1}^{n} \delta(x-x_m)\langle O_1(x_1)\cdots \delta O_m(x_m)\cdots O_n(x_n)\rangle, \quad (3)$$

where δO_m denotes the infinitesimal transformation of the local operator O_m. This shows that the correlation functions $\langle j^\mu(x)O_1(x_1)\cdots O_n(x_n)\rangle$ of the current j^μ must have the specific singularity at the coincident points, $x = x_m$, and this strongly constrains the possible forms of those correlators. Both the Nambu-Goldstone theorem and the anomaly matching condition can be obtained from this WT identity. This motivates us to generalize the notion of symmetry by maintaining the essential feature of the WT identity.

This can be done by noticing that the WT identity is equivalent to the topological nature of the Noether charge of closed $(d-1)$-manifolds [Gaiotto et al., 2015],

$$Q(M_{d-1}) = \int_{M_{d-1}} \star j, \quad (4)$$

where the notation of differential forms is used for later convenience, $j = g_{\mu\nu} j^\mu \mathrm{d}x^\nu$ and $\star j$ is the Hodge dual of j, i.e. $\star j = \frac{1}{(d-1)!}\varepsilon_{\mu\nu_1\cdots\nu_{d-1}} j^\mu \mathrm{d}x^{\nu_1}\wedge\cdots\wedge\mathrm{d}x^{\nu_{d-1}}$ on the flat space. When one takes M_{d-1} as the spatial slice of the constant time, $Q = \int j^0 \mathrm{d}^{d-1}x$, this coincides with the usual Noether charge.

The extended operator $Q(M_{d-1})$ is called topological if and only if the following operator identity holds

$$Q(M_{d-1}) = Q(M'_{d-1}), \quad (5)$$

where M_{d-1} and M'_{d-1} are related by some continuous deformations, and let us show it for the above Noether operator. The existence of a continuous deformation implies the presence of the d-dimensional volume V_d such that $\partial V_d = M_{d-1}\cup(-M'_{d-1})$, where $-M'_{d-1}$ is the orientation reverse of M'_{d-1}. As a result, one finds

$$\begin{aligned} Q(M_{d-1}) - Q(M'_{d-1}) &= \int_{M_{d-1}\cup(-M'_{d-1})} \star j \\ &= \int_{V_d} \mathrm{d}\star j \\ &= 0. \end{aligned} \quad (6)$$

Stokes' theorem is used to obtain the second line, and the last equality follows from the Noether conservation law as $\mathrm{d}\star j \propto \partial_\mu j^\mu \mathrm{d}^d x$. It would be obvious that one can reverse the logic: When the Noether operator, $Q(M_{d-1}) = \int_{M_{d-1}} \star j$, is topological for any M_{d-1}, one obtains the local conservation law, $\partial_\mu j^\mu = 0$. In this sense, the essence of the conservation law or the WT identity is nothing but the topological nature of $Q(M_{d-1})$.

This allows us to generalize the WT identity so that it holds not only for continuous symmetry but discrete symmetry: A d-dim QFT has an internal symmetry G if and only if

- there exists a topological operator $U_g(M_{d-1})$ for $g \in G$ and codim-1 closed manifolds M_{d-1},
- their fusion rule obeys the group law, $U_{g_1}(M_{d-1})U_{g_2}(M_{d-1}) = U_{g_1g_2}(M_{d-1})$,
- and the local operators transform as the representation of G when $U_g(M_{d-1})$ surrounds it.

For the continuous symmetry, one can obtain such $U_g(M_{d-1})$ by taking the exponential of $Q(M_{d-1})$, $U_{e^{i\alpha}}(M_{d-1}) = \exp(i\alpha Q(M_{d-1}))$, with appropriate local counterterms.

We note that the above definition can also apply to the discrete symmetry, such as the $\mathbb{Z}_2$ symmetry of the Ising model or real ϕ^4 theory. To see this explicitly, let us consider the d-dim classical Ising model,

$$Z = \sum_{\{s_x\}} \exp\left(\sum_{\langle x,x' \rangle} J s_x s_{x'} \right), \tag{7}$$

where $s_x = \pm 1$ denotes the classical $\mathbb{Z}_2$ spin. This obviously has the $\mathbb{Z}_2$ global symmetry, $s_x \mapsto -s_x$. The question is how to construct the corresponding topological defect, and one can define it as the defect operator $U(M_{d-1})$: When the bond $\langle x,x' \rangle$ is cut by the codim-1 surface M_{d-1}, the sign of J is flipped in the Boltzmann weight, i.e.

$$J_{\langle x,x' \rangle} = \begin{cases} J & (\langle x,x' \rangle \text{ is not cut by } M_{d-1}), \\ -J & (\langle x,x' \rangle \text{ is cut by } M_{d-1}). \end{cases} \tag{8}$$

In general, when there exists a lattice regularization such that symmetry has the on-site action, one can construct the topological defect in the same manner for both continuous and discrete G. As the topological nature of $U_g(M_{d-1})$ gives the WT identity when G is continuous, one obtains its discrete version of the WT identity by having the $U_g(M_{d-1})$ for discrete groups G.

Generalized symmetry = Topological defects

In the above discussion, we have found that conventional symmetry gives topological defects: *Symmetry* $\Rightarrow$ *Topological defects*. Then, it would be natural to ask if the converse is also true: *Toplogical defects* $\Rightarrow$ *Symmetry?*

The answer to this question is "No". When studying QFTs, one often encounters topological defects that are unrelated to any conventional symmetries. Even in such a situation, those topological defects play important roles as their topological feature implies conservation laws, and hence topological defects and conventional symmetries are equally useful. Nothing prevents us from enlarging our notion of symmetry, so one can think of topological defects as generalized symmetries.

In this regard, the conventional symmetry is specified by the codim-1 topological defects that obey the group multiplication. Among the three keywords, "codim-1", "topological", and "group multiplication", the topologicalness is the most important condition for symmetry. The new kind of "symmetry" can be obtained by replacing the other conditions. One direction of generalization is to consider topological defects defined on general codimension:

- **p-form symmetry** [Gaiotto et al., 2015]: Topological defects are defined on the codim-$(p+1)$ closed manifolds, $U_g(M_{d-p-1})$. Charged operators are defined on p-dim closed manifolds, $W(C_p)$.
  - When $p=0$, this reduces to the conventional symmetry.
  - When $p=1$, the charged operators are line operators. This includes the center symmetry of gauge theories as special cases.
- **n-group symmetry** [Sharpe, 2015, Benini et al., 2019, Cordova et al., 2019, Tanizaki and Unsal, 2020, Hidaka et al., 2021a,b]: (Nontrivial) mixture of p-form symmetries with $p=0,1,\ldots,n-1$.

Another kind of generalization is obtained by replacing the group-like fusion rule with more general ones:

- **Non-invertible symmetry** (or **categorical symmetry**): Let T_a be topological defects then the fusion rule is $T_a(M_{d-1})T_b(M_{d-1}) = \sum_c N_{ab}^c(M_{d-1})T_c(M_{d-1})$, where $N_{ab}^c(M_{d-1})$ is a c-number that can topologically depend on M_{d-1}.
  - When $a,b,c,\ldots$ lives in G and $N_{ab}^c = 1$ for $c = a\cdot b$ and $N_{ab}^c = 0$ for others, it reduces to the ordinary group-like symmetry.
  - In general, when one has a topological defect T_a, its inverse does not necessarily exist. This is why it is called "non-invertible".
  - In 2d QFTs, the finitely generated non-invertible symmetry is mathematically characterized by the fusion category, and the general theory is well developed [Bhardwaj and Tachikawa, 2018, Thorngren and Wang, 2019, Komargodski et al., 2021]. Therefore, this class of symmetry is also called "categorical".
  - In higher-dimensions, people just started to observe some examples of non-invertible symmetries [Nguyen et al., 2021, Koide et al., 2022, Choi et al., 2022, Kaidi et al., 2022, Hayashi and Tanizaki, 2022]. An interesting observation is that the fusion coefficient N_{ab}^c becomes the partition function of $(d-1)$-dim topological QFTs instead of just a number.
  - There also exists the non-invertible higher-form symmetry [Rudelius and Shao, 2020, Nguyen et al., 2021].

The Hilbert space of QFT should form a representation of generalized symmetry, so they provide the selection rule. The notion of spontaneous breaking is also well defined, and thus one can make the distinction between phases of matters by applying Landau's criterion. Furthermore, these symmetries can be gauged as in the case of ordinary symmetries. When there is an obstruction for the gauging procedure, it

provides important RG-invariant information as in the case of the anomaly matching condition.

Confinement-deconfinement phases and 1-form symmetry

In gauge theories, confinement/deconfinement of test charges is the characteristic feature that discriminates gapped phases [Wilson, 1974, 't Hooft, 1978]. In the context of finite-temperature gauge theories, this criterion is rephrased as the behavior of the Polyakov loop, or the Wilson loop wrapping along the imaginary-time circle. Since the finite-temperature pure gauge theories enjoy the symmetry acting on Polyakov loops, it has been known that the finite-temperature confinement-deconfinement criterion reduces to standard Landau's criterion of spontaneous breaking, and this symmetry has been called the center symmetry.

However, I would claim that this "center symmetry" was quite mysterious. It suddenly appears if we compactify the spacetime to a torus, but it does not exist for the noncompact spacetime $\mathbb{R}^d$ or the compact spacetime without nontrivial cycles such as the sphere S^d. Although we are dealing with the "local" QFTs, should we care about the "global" structure of the spacetime on which QFT lives to identify the symmetry?

It is the 1-form symmetry that resolves this uneasiness. To be specific, let us consider the d-dim pure $\mathrm{SU}(N)$ Yang-Mills theory. The Yang-Mills (YM) action is

$$S[a] = \frac{1}{g^2}\int \mathrm{tr}[F\wedge\star F], \tag{9}$$

where $F = \mathrm{d}a + ia\wedge a$ is the field strength of the $\mathrm{SU}(N)$ gauge field $a = a_{i,\mu}T^i\mathrm{d}x^\mu$. We shall see that there exists the codim-2 topological defect $U_k(M_{d-2})$ that obeys the $\mathbb{Z}_N$ fusion rule [Kapustin and Seiberg, 2014, Gaiotto et al., 2015],

$$U_k(M_{d-2})U_{k'}(M_{d-2}) = U_{k+k'}(M_{d-2}), \quad U_N(M_{d-2}) = 1, \tag{10}$$

and it can detect the N-ality of the Wilson line,

$$U_k(M_{d-2})W_R(C) = \exp\left(\frac{2\pi i}{N}k|R|\mathrm{link}(M_{d-2},C)\right)W_R(C), \tag{11}$$

where $W_R(C) = \mathrm{tr}_R[\exp(i\int_C a)]$ denotes the Wilson loop of the representation R, $|R|$ denotes its N-ality, and $\mathrm{link}(M_{d-2},C)$ is the linking number between M_{d-2} and C.

The easiest way to construct $U_k(M_{d-2})$ is to consider Wilson's lattice gauge formulation (see Appendix A of [Tanizaki and Ünsal, 2022]),

$$S[U_\ell] = K\sum_p\left(\mathrm{tr}\mathscr{P}\prod_{\ell\in p}U_\ell + \mathrm{c.c.}\right), \tag{12}$$

where $U_\ell \in \mathrm{SU}(N)$ is the link variable, p denotes each plaquette and $\mathscr{P}\prod_{\ell\in p}$ denotes the path-ordered product of the link variables around the plaquette p. In the case of the Ising model, one replaces the hopping term, or the nearest-neighbor coupling J, as in (8), and one can use the same trick here. That is, one should multiply $\mathbb{Z}_N$ phases B_p to the (inverse) coupling K that depend on plaquettes, and it defines

$$S[U_\ell, B_p] = K\sum_p \left(\mathrm{e}^{\frac{2\pi i}{N}B_p} \mathrm{tr}\mathscr{P}\prod_{\ell\in p} U_\ell + \mathrm{c.c.} \right), \tag{13}$$

and

$$B_p = \begin{cases} 0 & (p \text{ is not pierced by } M_{d-2}), \\ k & (p \text{ is pierced by } M_{d-2}). \end{cases} \tag{14}$$

Due to the closedness of M_{d-2}, B_p satisfies the flatness condition, that is, one can check $\exp(\frac{2\pi i}{N}\prod_{p\in c} B_p) = 1$ for any cubes c. Correlation functions that includes $U_k(M_{d-2})$ are then computed as

$$\langle U_k(M_{d-2})\cdots\rangle = \frac{1}{Z}\int \mathscr{D}U_\ell \exp(S[U_\ell, B_p])\cdots, \tag{15}$$

where $\cdots$ refers to other operator insertions. One can readily check that $U_k(M_{d-2})$ has the desired topological nature and detects the N-ality of test charges: Especially, the topological nature comes out of the redefinition of link variables by $\mathbb{Z}_N$ phase rotations. This confirms the $\mathbb{Z}_N$ 1-form symmetry of the pure $\mathrm{SU}(N)$ YM theory, which we denote as $\mathbb{Z}_N^{[1]}$.

The spontaneous breakdown of $\mathbb{Z}_N^{[1]}$ can be defined as the natural generalization of the off-diagonal long-range order (ODLRO), and it coincides with Wilson's criterion of quark confinement for this case. That is,

- $W_{\mathrm{fd}}(C)$ obeys the area law $\Leftrightarrow$ $\mathbb{Z}_N^{[1]}$ is unbroken.
- $W_{\mathrm{fd}}(C)$ obeys the perimeter law $\Leftrightarrow$ $\mathbb{Z}_N^{[1]}$ is spontaneously broken.

Here, we only consider the case of the fundamental Wilson loop for simplicity, but one can discuss the spontaneous breaking to a nontrivial subgroup by considering the ones with higher-dimensional representations.

Fradkin-Shenker's (non-)complementarity

It has been known that the Abelian-Higgs model has three phases, Coulomb, confinement, and Higgs phases. When the Higgs particle has the minimal electric charge, the confinement and Higgs phases are connected smoothly, while if the Higgs particle has a larger $U(1)$ charge those gapped phases are separated by the phase transition [Fradkin and Shenker, 1979, Banks and Rabinovici, 1979]. The continuity between confinement and Higgs regimes for the minimal $U(1)$ charge is often referred to as Fradkin-Shenker's complementarity.

Let us reinterpret this phenomenon from the viewpoint of the 1-form symmetry. For this purpose, let us quickly review the results of [Fradkin and Shenker, 1979]. The lattice model considered is given by

$$S = \beta \sum_{x,\mu} \cos(\partial_\mu \theta + N a_\mu) + K \sum_p \cos(f_{\mu\nu}), \tag{16}$$

where $\exp(i a_\mu(x)) \in U(x)$ is the $U(1)$ link variable, $\phi(x) = \exp(i\theta(x))$ is the Higgs field with the strict constraint $|\phi(x)| = 1$. Here, the spacetime dimension is 4. One can formally write the continuum theory as

$$S = \int \mathrm{d}^4 x \left(\frac{1}{2e^2} |f_{\mu\nu}|^2 + (|(\partial_\mu + iNa_\mu)\phi|^2 + g(|\phi|^2 - v^2)^2 \right) + (\text{monopoles}), \tag{17}$$

with $K \sim \frac{1}{e^2}$, $\beta \sim v^2$. In the strong coupling $K \ll 1$, monopoles are proliferated and one can naturally expect confinement of charges. On the other hand, in the weak couplings with the deep Higgs potential, $K \gg 1$ and $\beta \gg 1$, the system is in a Higgs phase, i.e. it is gapped and the electric flux is screened. When $K \gg 1$ and $\beta \ll 1$, nothing condenses and the system is in a Coulomb phase.

When $K = 0$ or $\beta = \infty$, the model enjoys significant simplification, and one can precisely evaluate the partition function. As a result, when $N = 1$, the confinement and Higgs regimes turn out to be smoothly connected. For $N \geq 2$, they are separated even in this limit, and it is a natural guess that the phase transition always separates these two regimes.

It can be checked that the theory (12) enjoys the $\mathbb{Z}_N^{[1]}$ symmetry, and its order parameters are Wilson loops. One can easily check its presence by repeating the discussion for the pure YM theory, but let us take another perspective here. Consider the 4d pure Maxwell theory, $S = \int \frac{1}{2e^2} f \wedge \star f$, then the Maxwell equation becomes

$$\mathrm{d}\left(\star \frac{1}{e^2} f \right) = 0, \quad \mathrm{d}\left(\frac{1}{2\pi} f \right) = 0. \tag{18}$$

The first one is the equation of motion, while the second one is the Bianchi identity. As both equations are written in the form of the "conservation law", one can define the topological operators,

$$U_\alpha^{(e)}(M_2) = \exp\left(i\alpha \int_{M_2} \frac{1}{e^2} \star f \right), \quad U_{\alpha'}^{(m)}(M_2) = \exp\left(i\alpha' \frac{1}{2\pi} \int_{M_2} f \right). \tag{19}$$

These are generators of the electric and magnetic 1-form symmetries that act on Wilson and 't Hooft loops, respectively. When one introduces the matter fields, Maxwell equations (18) are modified and these continuous symmetries are explicitly broken. The magnetic symmetry is completely broken due to the presence of lattice monopoles, but the electric one still maintains the $\mathbb{Z}_N^{[1]}$ symmetry as the charge of the Higgs particle is quantized to N in the unit of the elementary charge.

In the confinement regime, $\mathbb{Z}_N^{[1]}$ is unbroken as the Wilson loops obey the area law, while it is spontaneously broken in the Higgs regime. When $N \geq 2$, this shows that confinement and Higgs regimes should be separated by phase transitions. On the other hand, when $N = 1$, $\mathbb{Z}_1$ is a trivial group, and thus no 1-form symmetry discriminates the confinement and Higgs regimes.

When the system is gapped and $\mathbb{Z}_N^{[1]}$ is spontaneously broken, the low-energy field theory should contain the $\mathbb{Z}_N$ topological QFT and thus the ground state is described by a topologically-ordered state. In this regard, Fradkin-Shenker's non-complementarity for $N \geq 2$ gives an example of quantum phase transition between topologically trivial and non-trivial states.

Gauging of generalized symmetry

When a QFT enjoys the continuous symmetry generated by the Noether current j^μ, one can introduce a gauge field A_μ and consider the partition function,

$$Z[A] = \int \mathscr{D}\phi \exp\left(S[\phi] + i \int A_\mu j^\mu + \text{counter terms} \right). \tag{20}$$

This can be regarded as the generating functional for correlation functions of Noether current and this procedure is called background gauging. Naively, this partition function is invariant under the gauge transformation of A: Since Noether's conservation law implies

$$\int (A_\mu + \partial_\mu \lambda)\, j^\mu = \int A_\mu j^\mu - \int \lambda\, \partial_\mu j^\mu = \int A_\mu j^\mu, \tag{21}$$

one would obtain

$$Z[A + \mathrm{d}\lambda] = Z[A]. \tag{22}$$

When this gauge invariance actually holds, one can perform the path integral for gauge fields A, and the gauge theory coupled to the original QFT is obtained.

We note that this procedure is not restricted to continuous symmetry: It can be performed for discrete symmetry, higher-form, and n-group symmetries, and even for non-invertible symmetry. To get an idea about how this can be done, let us consider the case of the ordinary discrete symmetry G, in particular, the $\mathbb{Z}_2$ symmetry of the Ising model (7). Since this is already a lattice theory, it is natural to introduce the $\mathbb{Z}_2$ link variable $U_\ell = U_{\langle x,x' \rangle} \in \{\pm 1\}$, and one has

$$Z[\{U_\ell\}] = \sum_{\{s_x\}} \exp\left(J \sum_{\langle x,x' \rangle} s_x U_{\langle x,x' \rangle} s_{x'} \right). \tag{23}$$

As a result, the $\mathbb{Z}_2$ symmetry is promoted to the local gauge redundancy:

$$s_x \mapsto (-1)^{\lambda_x} s_x, \quad U_{\langle x,x'\rangle} \mapsto (-1)^{\lambda_x+\lambda_{x'}} U_{\langle x,x'\rangle}. \tag{24}$$

One needs to take the continuum limit to find the relativistic QFT from the lattice model. This operation is usually done in the weak-coupling regime at the lattice scale, so it is natural to require that the discrete gauge fields have to be flat:

$$\prod_{\ell\in p} U_\ell = 1 \in \mathbb{Z}_2. \tag{25}$$

By comparing this with the definition of the $\mathbb{Z}_2$ symmetry defects $U(M_{d-1})$ given in (8), $\{U_\ell\}$ with the flatness condition is nothing but the specific network of the $\mathbb{Z}_2$ symmetry defects.

In general, the (flat) gauge field A for the discrete symmetry G can be identified as the network of the symmetry defects $U_g(M_{d-1})$. Its gauge transformation recombines the given network, and the gauge invariance requires that the partition function $Z[A]$ is invariant under any recombination. This idea has a straightforward generalization to any kind of symmetries: For the case of 1-form symmetry, the background gauge field is the same with a network of the codim-2 topological defects. When the gauge invariance holds, all the possible networks can be summed consistently, and one obtains the dynamically gauged model.

't Hooft anomaly and anomaly matching

Gauging the symmetry is useful not only to construct new QFTs starting from a given one but also for studying the dynamics of the original QFT. Especially, the violation of background gauge invariance provides the RG-invariant data ['t Hooft, 1980], and it constrains the possible dynamics of strongly-coupled QFTs.

Let us consider a d-dim QFT with the symmetry G. Here, G can refer to the higher-group, and let us formally denote its background gauge field as A. If the partition function has the following type of anomaly, it is called the 't Hooft anomaly of G [Kapustin and Thorngren, 2014, Wen, 2013]:

$$Z[A+\delta\lambda] = \exp(i\mathscr{A}[A,\lambda])Z[A], \tag{26}$$

where $\mathscr{A}[A,\lambda]$ is a d-dim local functional of the gauge field A and its gauge parameter λ that cannot be eliminated by adding local counterterms.

Empirically, when a d-dim QFT has an 't Hooft anomaly, there always exists a classical $(d+1)$-dim topological G-gauge theory $S_{d+1}[A]$ that reproduces the same anomaly on the boundary. This is the generalization of Callan-Harvey's anomaly inflow [Callan and Harvey, 1985]: When a d-dim QFT is defined on M_d, let us take a $(d+1)$-manifold N_{d+1} such that $\partial N_{d+1} = M_d$, and we also extend the G-gauge field A on M_d to the one on N_{d+1}. By regarding our d-dim QFT as the boundary of the $(d+1)$-dim classical topological theory, the following partition function,

$$Z_{M_d}[A]\exp(iS_{d+1,N_{d+1}}[A]), \tag{27}$$

is fully gauge invariant.

In the modern condensed-matter physics, this $(d+1)$-dim classical topological action is regarded as the partition function of a $(d+1)$-dimensional symmetry-protected topological (SPT) state with the symmetry G. Unlike the intrinsic topological orders, the SPT states do not have anyons and, moreover, the ground state is always unique for any closed spatial manifolds. This is indeed almost trivial as it can be continuously connected to the trivial gapped vacuum when arbitrary (including G-breaking) perturbations are introduced. Still, it is nontrivial as one cannot reach the trivial gapped state without encountering quantum phase transitions as long as the symmetry G is preserved.

This nontriviality can be understood by creating the boundary, then the symmetry G is realized in an anomalous way. This is the d-dim 't Hooft anomaly discussed above, and there must be some boundary degrees of freedom that cancels the anomaly to be consistent with the gauge invariance. This is the physical picture of the anomaly-inflow argument.

This physical picture is helpful to have a clear understanding of the anomaly matching condition. Let us consider the $(d+1)$-dim G-SPT with the boundary and also assume the boundary condition is carefully chosen so that the d-dim QFT of our interest is realized on the boundary. We then take the low-energy limit to perform the nonperturbative RG transformation. Since the $(d+1)$-dim SPT is uniquely gapped, nothing changes under the RG flow so the RG transformation acts nontrivially only on the boundary theory. However, the full G-gauge invariance is the exact feature of the system, and thus the d-dim low-energy theory should cancel the same anomaly inflow from the bulk. This clearly tells the anomaly matching between the ultraviolet (UV) and infrared (IR) theories,

$$\mathscr{A}_{\mathrm{UV}}[A,\lambda] = \mathscr{A}_{\mathrm{IR}}[A,\lambda]. \tag{28}$$

Especially, the presence of 't Hooft anomaly shows that the IR theory must be nontrivial since QFTs with the unique, gapped ground state cannot have nontrivial 't Hooft anomalies. In the relativistic context, the low-energy effective theory must contain gapless excitations, topological degrees of freedom, and/or spontaneous breakdown of G to match the nontrivial anomaly.

The correspondence between d-dim anomaly and $(d+1)$-dim SPT states also provides a useful tool to classify the possible form of anomalies. Because of the simplicity of the SPT state, its classification is now promoted to the rigorous mathematical theory with the help of the algebraic topology [Kapustin, 2014, Freed and Hopkins, 2021, Yonekura, 2019a]. This provides a useful tool to understand the nonperturbative anomalies that have been missed in conventional techniques.

Physics of θ angle: Application of the $\mathbb{Z}_N^{[1]}$ symmetry for YM theory

In 4d SU(N) YM theory, the renormalizable terms consist of the YM kinetic term and the topological term, and one gets the θ vacua by summing up all the topological sectors of $\pi_3(\mathrm{SU}(N)) \simeq \mathbb{Z}$,

$$Z_\theta = \int \mathscr{D}a \exp\left(-\frac{1}{g^2} \int \mathrm{tr}[F \wedge \star F] + i\frac{\theta}{8\pi^2} \int \mathrm{tr}[F \wedge F] \right). \tag{29}$$

The effect of the θ term on low-energy physics is quite nontrivial and a difficult problem. In the 't Hooft large-N limit, $N \to \infty$ with fixed g^2N, the θ dependence of the ground-state energy density can be computed as [Witten, 1980]:

$$E(\theta) - E(0) = -\frac{1}{\mathrm{vol}} \ln(Z_\theta / Z_0) = \frac{\chi_{\mathrm{top}}}{2} \theta^2 + O(\theta^4/N^2). \tag{30}$$

However, the partition function must satisfy the 2π periodicity by definition of the topological charge, $Z_{\theta+2\pi} = Z_\theta$, and thus this cannot be the complete story as this expression violates the 2π periodicity. If there are meta-stable YM vacua whose lifetimes are exponentially long in large-N, this issue can be resolved by considering the multi-branch structure [Witten, 1980]:

$$E(\theta) = \min_{k \in \mathbb{Z}} E_k(\theta), \quad E_k(\theta) = E_0(\theta - 2\pi k). \tag{31}$$

This suggests that the YM theory has two vacua at $\theta = \pi$ as a result of spontaneous *CP* breaking.

We can get an intuitive scenario for the multi-branch vacua by taking an Abelian gauge ['t Hooft, 1981]. In such a gauge, the resultant $U(1)$ gauge fields have various singularities that can be regarded as monopoles, vortices, etc. and one can think of the confining vacuum as the result of proliferated monopoles. However, the particle-like spectrum contains not only monopoles but also dyons, which have both electric and magnetic charges. When one turns on the θ angle, the magnetically charged particles acquire the fractional electric charge due to the Witten effect [Witten, 1979], and the role of monopole and dyon should be exchanged when one dials $\theta : 0 \to 2\pi$. Therefore, by regarding the original vacuum as the monopole condensing state and another vacuum as the dyon condensing state, one can naturally interpret the above level-crossing phenomenon as the first-order quantum phase transition between these states.

Although this explanation is intuitive and tells a lot of physics behind it, it is unclear if the phase transition is really there. Since both monopoles and dyons are magnetically charged particles, the Wilson loop obeys the area law for both condensations. This seems to imply that the vacua at $\theta \sim 0$ and $\theta \sim 2\pi$ belong to the same phase, and then it is reasonable to ask if there is a continuous path in the space of couplings that smoothly connect these vacua. Moreover, if such a path exists in the

coupling space, we would have no way to exclude that it is the θ parameter as the actual dynamics is strongly coupled.

When one assumes confinement, it can be rigorously proven that the vacua at θ and $\theta + 2\pi$ should be distinct as the $\mathbb{Z}_N^{[1]}$-SPT states [Gaiotto et al., 2017]. Since the pure $\mathrm{SU}(N)$ YM theory has the $\mathbb{Z}_N^{[1]}$ symmetry, we can introduce its background gauge field. As we have discussed earlier, the background gauge field for $\mathbb{Z}_N^{[1]}$ is nothing but the network of codim-2 topological defects, and it is equivalent to the plaquette $\mathbb{Z}_N$ gauge field, B_p, in (13). Let us denote it as B formally and call it the $\mathbb{Z}_N$ two-form gauge field because the Poincare dual of the codim-2 surface is a two-form.

When a codim-2 surface in the network wraps a nontrivial 2-cycle of the spacetime, it is equivalent to take the 't Hooft twisted boundary condition ['t Hooft, 1979] along that cycle. Under such a twisted boundary condition, the topological charge is fractionally shifted [van Baal, 1982]:

$$\int \frac{1}{8\pi^2}\mathrm{tr}[F\wedge F] \in \underbrace{\frac{N}{8\pi^2}\int B\wedge B}_{\in(1/N)\mathbb{Z}} + \mathbb{Z}. \tag{32}$$

As a result, one obtains the following exact relation for the YM partition function [Gaiotto et al., 2017],

$$Z_{\theta+2\pi}[B] = \exp\left(i\frac{N}{4\pi}\int B\wedge B\right) Z_\theta[B]. \tag{33}$$

This can be regarded as a generalization of anomaly and it requires the presence of the quantum phase transition when θ is changed by 2π (For more detailed discussions, see also [Tanizaki and Kikuchi, 2017, Kikuchi and Tanizaki, 2017, Tanizaki and Sulejmanpasic, 2018, Karasik and Komargodski, 2019, Cordova et al., 2020a,b]). Two confining states given by the monopole and dyon condensation have different partition functions when one introduces the background gauge field B by the contact term $\sim \frac{N}{4\pi}B\wedge B$, so they are different as $\mathbb{Z}_N^{[1]}$-SPT states.

Let us assume that CP at $\theta = \pi$ is spontaneously broken to satisfy the anomaly matching condition. This allows us to introduce the CP domain wall that connects these two vacua. Since these vacua are different $\mathbb{Z}_N^{[1]}$-SPT states, the anomaly-inflow argument predicts the $\mathrm{SU}(N)_1$ Chern-Simons theory on the boundary [Gaiotto et al., 2017]. Although the test quark is assumed to be confined on the bulk, the deconfinement necessarily occurs on the wall (see also [Sulejmanpasic et al., 2017, Komargodski et al., 2018]).

Applications to 4d QCD with fundamental quarks

As generalized symmetry has a beautiful application to the physics of pure YM theory, it is natural to ask if one can apply it to the case of QCD with fundamental quarks. Before that, the following question needs to be considered: Is there any generalized symmetries in QCD with fundamental matters?

In the case of QCD, the new kind of symmetry has not been found, unfortunately. Despite this fact, many new anomalies are recently found for QCD in the chiral limit using the technique of 1-form gauging [Tanizaki and Kikuchi, 2017, Shimizu and Yonekura, 2018, Gaiotto et al., 2018, Tanizaki et al., 2017, 2018, Tanizaki, 2018, Yonekura, 2019b]. Without having 1-form symmetries, its technique turns out to be useful to obtain the refined information of the conventional symmetry.

Why are new anomalies found without having new symmetries? The key point is the correct identification of the symmetry group. When one writes down the chiral symmetry of massless QCD, it is often written as $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$, where N_f is the number of massless flavors. However, since the vector-like symmetry should be $U(N_f)_V \simeq (SU(N_f)_{\text{diag}} \times U(1)_V)/\mathbb{Z}_{N_f}$ instead of $SU(N_f)_{\text{diag}} \times U(1)_V$, one should divide the above group by the common $\mathbb{Z}_{N_f}$ center and the symmetry group becomes $[SU(N_f)_L \times SU(N_f)_R \times U(1)_V]/\mathbb{Z}_{N_f}$. This is, however, not yet complete. Since the global symmetry must act non-trivially on physical local operators, one must further divide the above group by the common center with the gauge group, and the correct symmetry group of massless QCD is given by

$$G = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V}{\mathbb{Z}_{N_f} \times \mathbb{Z}_{N_c}}, \tag{34}$$

where N_c is the number of color.

To detect the 't Hooft anomaly, one should introduce the background G-gauge field, and here there exists a huge difference depending on whether one takes into account the correct quotient structure of G or not. One must introduce not only the 1-form gauge fields but also the 2-form gauge fields to properly gauge G, and the 2-form gauge fields play the pivotal role to find the new anomaly. For example, when one pays attention to the vector-like subgroup and the discrete axial symmetry,

$$\frac{SU(N_f)_{\text{diag}} \times U(1)_V}{\mathbb{Z}_{N_f} \times \mathbb{Z}_{N_c}} \times (\mathbb{Z}_{N_f})_\chi \subset G, \tag{35}$$

its background gauge field gives the anomalous violation of the baryon-number current [Tanizaki, 2018],

$$\partial_\mu J^\mu_{\text{baryon}} = \frac{N_f}{(2\pi)^2} B_f \wedge \mathrm{d}A_\chi, \tag{36}$$

where B_f is the $\mathbb{Z}_{N_f}$ two-form gauge field to describe the $\mathbb{Z}_{N_f}$ quotient and A_χ is the discrete chiral gauge field. We note that this anomaly vanishes if B_f is turned off so this baryon-number anomaly for the discrete chiral symmetry can be detected only if one takes into account the quotient structure.

Such a discrete anomaly of massless QCD has an interesting application to the QCD phase diagram with imaginary chemical potentials. In the case of the baryon imaginary chemical potential, the one-loop Polyakov-loop potential predicts that the high-temperature phase has the Roberge-Weiss phase transition [Roberge and Weiss, 1986]. At the Roberge-Weiss phase transition point, there exists the $\mathbb{Z}_2$ symmetry acting on the Polyakov loop, and one can properly define the deconfinement as the breaking of $\mathbb{Z}_2$ symmetry [Shimizu and Yonekura, 2018]. One can instead consider the case of flavor-dependent imaginary chemical potential [Kouno et al., 2012, Iritani et al., 2015], and then it can realize the full $\mathbb{Z}_N$ symmetry acting on the Polkyakov loop [Cherman et al., 2017]. In this way, the imaginary chemical potentials provide a playground of QCD with fundamental quarks with the well-defined confinement-deconfinement transition with the Polyakov-loop order parameter.

The discrete anomaly discussed above turns out to persist in the S^1-compactified spacetime with the suitable choice of the imaginary-chemical potentials [Shimizu and Yonekura, 2018, Tanizaki et al., 2017, 2018]. The anomaly matching condition then tells that phases with the special value of imaginary chemical potentials must be always nontrivial, and it predicts that the chiral restoration temperature must be above the deconfinement temperature:

$$T_{\text{chiral}} \geq T_{\text{deconf}}. \tag{37}$$

One can prove this inequality as long as N_c and N_f has a nontrivial common divisor, $\gcd(N_c, N_f) > 1$.

There exists an interesting consistency check with previous literature about this inequality. When one takes the 't Hooft-type large-N_c limit, i.e. $N_c \to \infty$ with fixed N_f, the effect of the quark boundary condition is just a subleading correction in large N_c. Therefore, the above inequality should be true for thermal large-N_c QCD, and then we can compare it with holographic QCD [Sakai and Sugimoto, 2005a,b]. This has been studied in [Aharony et al., 2007] using the D4-D8-$\overline{\text{D8}}$ model. In the standard Sakai-Sugimoto setup where the D8 and $\overline{\text{D8}}$ branes are set antipodally, the chiral restoration and deconfinement occur at the same temperature, $T_{\text{chiral}} = T_{\text{deconf}}$. When the separation of those flavor branes is changed, those two transition temperatures can take different values but it turns out that the chiral restoration always occurs after the deconfinement, $T_{\text{chiral}} > T_{\text{deconf}}$. In both cases, the result is consistent with the discrete anomaly matching condition.

Summary

We have seen that the notion of symmetry is drastically extended, and it provides a useful tool to analyze strongly-coupled QFTs. For this generalization, it is important to notice that the presence of topological defects implies the conservation law, and we can equally use it as we have used conventional symmetry. This extends the

systematic characterization of phases of matter by its spontaneous breaking, and we can also perform its gauging to find the new anomaly matching condition.

In the case of QCD with fundamental quarks, we have currently found no generalized symmetries. Despite this fact, the techniques developed for higher symmetries turn out to be useful for getting more information about conventional chiral symmetry. The correct identification of the global symmetry provides the new nonperturbative anomaly, and it gives an interesting constraint on the phase diagram with imaginary chemical potentials.

Lastly, it should be emphasized that the scope of generalized symmetries is still rapidly growing. Honestly, I am quite astonished by the fact that symmetries of QFTs are so fruitful and we newly discover them in this 21st century. It is my great pleasure if this review gives a useful introduction for readers to this exciting and developing research field.

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